MIXED PROBLEMS ON TORSION OF AN ELASTIC HALF-SPACE WITH A SPHERICAL INCLUSION

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A.N. RUKHOVETS and Ia.S. UFLIAND
(Leningrad)

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Using the method of dual series, solutions are obtained to some problems on the torsion, by means of a circular punch, of a half-space with a spherical inclusion.

The cases of a rigid spherical inclusion and of a spherical cavity are studied. In both instances, the problem is first reduced to the associated Legendre polynomials and then to a Fredholm integral equation. The effective solution to the problem is obtained by a series expansion in terms of a small parameter which relates the radius of the cavity to the distance from its center to the half-space boundary.

The relations between the angle of twist of the punch and the applied torsional moment are found.

1. Consider a half-space which is attached to a rigid punch and to a fixed spherical inclusion, and which is subjected to a torque M applied via the punch. If we assume that the angle of twist of the punch is ϑ, then the stress-deformation state is defined by the function $v(r, z)$ satisfying the differential Eq.

$$\Delta v - r^{-2} v = 0 \qquad (z > 0) \tag{1.1}$$

and the boundary conditions

$$v|_S = \vartheta r, \qquad \left.\frac{\partial v}{\partial z}\right|_{S'} = 0, \qquad v|_\Sigma = 0 \tag{1.2}$$

Here S and S^1 are respectively, the region under the punch and the region exterior to the punch on the plane $z = 0$ while Σ is the surface of the sphere.

In seeking a solution, it is convenient to introduce the bispherical coordinate system (α, β, ϕ) defined by Formulas (Fig. 1)

$$x = \frac{a \sin\alpha \cos\varphi}{\operatorname{ch}\beta - \cos\alpha},$$

$$y = \frac{a \sin\alpha \sin\varphi}{\operatorname{ch}\beta - \cos\alpha}, \qquad z = \frac{a \operatorname{sh}\beta}{\operatorname{ch}\beta - \cos\alpha} \tag{1.3}$$

$$(0 \leqslant \beta \leqslant \beta_0,\ 0 \leqslant \alpha \leqslant \pi,\ -\pi \leqslant \varphi \leqslant \pi)$$

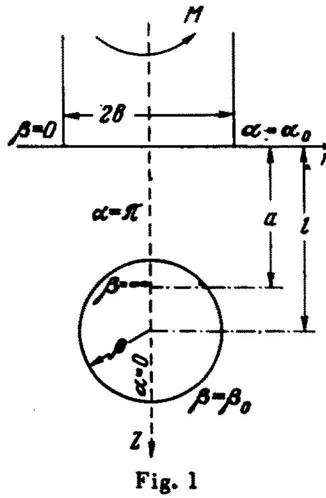

Fig. 1

Then separation of variables in (1.1) and application of the last condition in (1.2) yields

$$v(\alpha, \beta) = \sqrt{2}\sqrt{\operatorname{ch}\beta - \cos\alpha}\sum_{n=1}^{\infty} A_n \operatorname{sh}(n + 1/2)(\beta_0 - \beta) P_n^1(\cos\alpha) \tag{1.4}$$

Rewriting the remaining boundary conditions in (1.2) in the form

$$v|_{\beta=0} = \vartheta \frac{a \sin\alpha}{1 - \cos\alpha} \qquad (\alpha_0 < \alpha \leqslant \pi) \tag{1.5}$$

$$\left.\frac{\partial v}{\partial \beta}\right|_{\beta=0} = 0 \qquad (0 \leqslant \alpha < \alpha_0) \tag{1.6}$$

we obtain the following dual series Eqs. (*):

$$\sum_{n=1}^{\infty} A_n \operatorname{sh}(n + {}^1/_2)\,\beta_0 P_n{}^1(\cos\alpha) = \frac{a\vartheta \sin\alpha}{\sqrt{2}\,(1-\cos\alpha)^{3/2}} \qquad (\alpha_0 < \alpha \leqslant \pi) \tag{1.7}$$

$$\sum_{n=1}^{\infty} (n + {}^1/_2)\, A_n \operatorname{ch}(n + {}^1/_2)\,\beta_0\, P_n^1(\cos\alpha) = 0 \qquad (0 \leqslant \alpha < \alpha_0) \tag{1.8}$$

Set

$$A_n \operatorname{ch}(n + {}^1/_2)\,\beta_0 = \int_{\alpha_0}^{\pi} \varphi(t) \sin(n + {}^1/_2)\, t\, dt \tag{1.9}$$

Then (1.8) is satisfied as a result of the relations [11]

$$P_n{}^1(\lambda) = -\sqrt{1-\lambda^2}\,\frac{d}{d\lambda} P_n(\lambda), \qquad \lambda = \cos\alpha \tag{1.10}$$

$$\sum_{n=0}^{\infty} \cos\left(n + \frac{1}{2}\right) t P_n(\cos\alpha) = 0, \qquad t > \alpha \tag{1.11}$$

The second of the dual series relations (1.7) may be transformed with the aid of Formulas [12]

$$\sum_{n=0}^{\infty} \sin\left(n + \frac{1}{2}\right) t P_n(\cos\alpha) \begin{cases} 0, & t < \alpha \\ [2(\cos\alpha - \cos t)]^{-1/2}, & t > \alpha \end{cases} \tag{1.12}$$

$$P_n(\cos\alpha) = \frac{\sqrt{2}}{\pi} \int_{\alpha}^{\pi} \frac{\sin\left(n + \frac{1}{2}\right) x}{\sqrt{\cos\alpha - \cos x}}\, dx \tag{1.13}$$

into the form

$$\frac{d}{d\lambda} \int_{\alpha}^{\pi} \frac{dx}{\sqrt{\cos\alpha - \cos x}} \left\{\varphi(x) - \frac{2}{\pi} \int_{\alpha_0}^{\pi} \varphi(t)\,[\eta(t-x) - \eta(t+x)]\, dt\right\} = -\frac{\vartheta a}{(1-\lambda)^{3/2}} \tag{1.14}$$

$$\eta(u) = \sum_{n=0}^{\infty} \frac{\cos(n + {}^1/_2)\, u}{\exp[2(n + {}^1/_2)\,\beta_0] + 1} \tag{1.15}$$

Integrating (1.14) with respect to λ, we obtain Abel's integral Eq.

$$\int_{\alpha}^{\pi} \frac{dx}{\sqrt{\cos\alpha - \cos x}} \left\{\varphi(x) - \frac{2}{\pi} \int_{\alpha_0}^{\pi} \varphi(t)[\eta(t-x) - \eta(t+x)]\,dt\right\} = -\frac{2\vartheta a}{\sqrt{1-\cos a}} + c_0 \tag{1.16}$$

The solution of this equation may be reduced, after some manipulation, to a Fredholm integral equation in $\phi(x)$

$$\varphi(x) - \frac{2}{\pi} \int_{\alpha_0}^{\pi} \varphi(t)\,[\eta(t-x) - \eta(t+x)]\, dt =$$
$$= \frac{c_0}{\pi} \sqrt{2} \sin\frac{x}{2} - \frac{2}{\pi}\,\vartheta a\, \frac{1}{\sin{}^1/_2\, x} \qquad (\alpha_0 < x < \pi) \tag{1.17}$$

The constant c_0 in the right-hand side of the equation is determined from the supplemen-

*) Dual series of the functions $P^m{}_n$ $(\cos\alpha)$ for the cases of $m = 0$ and $m = 1$ are examined in detail in [1 to 6]. The case of arbitrary m is examined in [7 to 10].

tary integrability condition of $\partial v/\partial\beta$ for $\beta = 0$ in the region $\alpha_0 < \alpha < \pi$ (This condition is equivalent to the requirement that the applied torque on the punch be finite). We now utilize (1.9) to (1.11) and Formula

$$\sum_{n=0}^{\infty} \cos (n + 1/2)\, tP_n (\cos\alpha) = \frac{1}{\sqrt{2(\cos t - \cos\alpha)}}, \qquad t < \alpha \tag{1.18}$$

The Expression

$$\frac{\partial v}{\partial\beta}\Big|_{\beta=0} = -\sqrt{2}\sqrt{1-\cos\alpha}\sum_{n=1}^{\infty} A_n (n + 1/2) \operatorname{ch} (n + 1/2)\beta_0 P_n^1 (\cos\alpha) \tag{1.19}$$

is reduced to the form

$$\frac{\partial v}{\partial\beta}\Big|_{\beta=0} = -\sqrt{1-\cos\alpha}\left[\varphi(\alpha_0)\frac{d}{d\alpha}\frac{1}{\sqrt{\cos\alpha_0 - \cos\alpha}} + \frac{d}{d\alpha}\int_{\alpha_0}^{\alpha}\frac{\varphi'(t)\,dt}{\sqrt{\cos t - \cos\alpha}}\right] \tag{1.20}$$

In order that the above mentioned condition be satisfied, it is clearly necessary to set in (1.20)

$$\varphi(\alpha_0) = 0 \tag{1.21}$$

This is the condition for the determination of c_0.

Now the moment M is easily found:

$$M = -G\iint_S \frac{\partial v}{\partial z}\Big|_{z=0,\ r<b} r^2\,dr\,d\varphi = -2\pi G\int_0^b \frac{\partial v}{\partial z}\Big|_{z=0} r^2\,dr \tag{1.22}$$

Here G is the shear modulus and b is the radius of the punch.

From (1.20) and (1.21) we have

$$\frac{\partial v}{\partial z}\Big|_{z=0} = \frac{1-\cos\alpha}{a}\frac{\partial v}{\partial\beta}\Big|_{\beta=0} = -\frac{1}{a}(1-\cos\alpha)^{3/2}\frac{d}{d\alpha}\int_{\alpha_0}^{\alpha}\frac{\varphi'(t)\,at}{\sqrt{\cos t - \cos\alpha}}$$

so that

$$M = 2\pi G a^2\int_{\alpha_0}^{\pi}\frac{\sin^2\alpha}{(1-\cos\alpha)^{1/2}}\frac{d}{d\alpha}\left(\int_{\alpha_0}^{\alpha}\frac{\varphi'(t)\,dt}{\sqrt{\cos t - \cos\alpha}}\right)d\alpha \tag{1.23}$$

Integrating by parts and inverting the order of integration, we obtain the relation between the moment M and the angle of twist ϑ

$$M = 2\pi G a^2\int_{\alpha_0}^{\pi}\frac{\varphi(t)}{\sin^3(t/2)}\,dt \tag{1.24}$$

Thus, the solution of the problem has been reduced to a Fredholm integral Eq. (1.17) which generally has to be solved by numerical methods. Here, there is an additional difficulty, since the kernel does not appear in explicit form. Nevertheless, for suitably small values of the ratio ρ/l, the method of successive approximations can be effectively applied.

As a preliminary step, expand the kernel in (1.17) in a power series $\varepsilon = \exp(-\beta_0)$

$$\eta(t-x) - \eta(t+x) = 2\,[(\varepsilon - \varepsilon^2 + \varepsilon^3 - \ldots)\sin {}^1/_2 t\sin {}^1/_2 x + \tag{1.25}$$
$$+ (\varepsilon^3 - \varepsilon^6 + \ldots)\sin {}^3/_2 t\sin {}^3/_2 x + (\varepsilon^5 - \varepsilon^{10} + \ldots)\sin {}^5/_2 t\sin {}^5/_2 x + \ldots]$$

The relations between the quantities α_0 and β_0 and the geometric parameters are given by

$$\frac{l}{\rho} = \operatorname{ch}\beta_0, \qquad \frac{l}{b} = \frac{\operatorname{cth}\beta_0}{\operatorname{ctg}{}^1/_2\alpha_0}$$

To conform with the above, let us also expand $\phi(x)$ and c_0 into similar series. Then (1.17) takes the form

$$\varphi_0(x)+\varphi_1(x)\varepsilon+\varphi_2(x)\varepsilon^2+\cdots=\frac{4}{\pi}\int_{\alpha_0}^{\pi}[\varphi_0(t)+\varphi_1(t)\varepsilon+\varphi_2(t)\varepsilon^2+\ldots]\times$$

$$\times\Big[(\varepsilon-\varepsilon^2+\varepsilon^3-\ldots)\sin\frac{t}{2}\sin\frac{x}{2}+(\varepsilon^3-\varepsilon^6+\ldots)\sin\frac{3t}{2}\sin\frac{3x}{2}+(\varepsilon^5-\varepsilon^{10}+\cdots)\times$$

$$\times\sin\frac{5t}{2}\sin\frac{5x}{2}+\cdots\Big]dt+\frac{\sqrt{2}}{\pi}\sin\frac{x}{2}(c_{00}+c_{01}\varepsilon+c_{02}\varepsilon^2+\ldots)-\frac{2va}{\pi}\frac{1}{\sin{}^1/_2\, x}\tag{1.26}$$

Equating coefficients of like powers of the parameter, we obtain for the zeroeth approximation

$$\varphi_0(x)=\frac{2\vartheta a}{\pi}\left(\frac{\sin{}^1/_2\, x}{(\sin{}^1/_2\,\alpha_0)^2}-\frac{1}{\sin{}^1/_2\, x}\right)\tag{1.27}$$

$$M_0=2\pi Ga^2\int_{\alpha_0}^{\pi}\frac{\varphi_0(t)}{(\sin{}^1/_2\, t)^3}dt=\frac{16}{3}G\vartheta b^3\tag{1.28}$$

Formula (1.28) coincides with the known expression for the torsional moment in the case of a continuous half-space.

Successive approximations yield

$$\varphi_1(x)=\varphi_2(x)=0,\quad \varphi_3(x)=16\vartheta a\pi^{-2}(\sin\alpha_0-\pi+\alpha_0)\sin{}^1/_2\, x\,(\cos x-\cos\alpha_0)\tag{1.29}$$

Consequently, the first correction to the moment M_0 is of third order of smallness, so that we have, with a good degree of accuracy

$$M\approx{}^{16}/_3\, G\vartheta b^3(1+\Delta),\quad \Delta=12\,\varepsilon^3\pi^{-1}(\mathrm{tg}\,{}^1/_2\,\alpha_0)^3(\sin\alpha_0-\pi+\alpha_0)^2\tag{1.30}$$

Values of the correction Δ for some values of ρ/l and b/l are given below.

$\rho/l=0.6$	0.7	0.8	
$\Delta=0.012$	0.031	0.082	$(b/l={}^1/_3)$
$\Delta=0.027$	0.059	0.133	$(b/l={}^1/_2)$
$\Delta=0.048$	0.085	0.147	$(b/l=1)$

The quantity $1+\Delta$ characterizes the magnification of the moment M associated with the effect of a rigid, stationary inclusion.

Note that the proposed method could also be used in the solution of a more general problem in which torsional moments M and M' are applied to both, the punch and the inclusion. Whereupon, two angles of rotation ϑ and ϑ' for the punch and the inclusion, respectively, must be introduced, so that the last condition in (1.2) is no longer homogeneous. Upon calculation of the moments M and M', we obtain a system of equations with the two unknowns ϑ and ϑ'. The case examined above corresponds to $\vartheta'=0$, so that calculations of M' may be based on the solution previously obtained.

2. We will now obtain the solution for the twisting by means of a round punch of a half-space with a spherical cavity whose surface is stress-free.

Instead of utilizing the stress function Φ which is generally used in problems of this type and which satisfies Eq. [13]

$$\Delta\Phi-\frac{4}{r}\frac{\partial\Phi}{\partial r}=0\tag{2.1}$$

it is convenient in this case to introduce another function $w(r, z)$ defined as

$$\Phi=r^2w\tag{2.2}$$

It is easily verified that w satisfies Eq.

$$\Delta w-4r^{-2}w=0\tag{2.3}$$

and, consequently, may be represented in bispherical coordinates by the series

$$w=\sqrt{2}\sqrt{\mathrm{ch}\,\beta-\cos\alpha}\sum_{n=2}^{\infty}[A_n\,\mathrm{sh}\,(n+{}^1/_2)(\beta_0-\beta)+B_n\,\mathrm{sh}\,(n+{}^1/_2)\beta]\,P_n^2(\cos\alpha)\tag{2.4}$$

Employing known relations from torsion theory [13], the boundary conditions for $w(\alpha, \beta)$ become

$$[\partial w / \partial \beta]_{\beta=0} = 0 \qquad (\alpha_0 < \alpha \leqslant \pi) \tag{2.5}$$

$$w \,|_{\beta=0} = C\,(1 - \cos\alpha)^2 \,|\, a^2 \sin^2\alpha \qquad (0 \leqslant \alpha < \alpha_0) \tag{2.6}$$

$$w \,|_{\beta=\beta_0} = 0 \qquad (0 \leqslant \alpha \leqslant \pi) \tag{2.7}$$

Formulating the expression for the moment due to the loading on a hemisphere of arbitrary radius with the center at the origin, we can obtain the following relation between C and the torsional moment M:

$$C = -{}^1/_2 M / \pi G \tag{2.8}$$

The condition (2.7) yields $B_n = 0$. Whereupon (2.5) and (2.6) lead to a system of dual series in A_n:

$$\sum_{n=2}^{\infty} A_n \,\mathrm{sh}\,(n + {}^1/_2)\,\beta_0 P_n^{\,2}(\cos\alpha) = \frac{C}{\sqrt{2}} \frac{(1-\cos\alpha)^{3/2}}{a^2 \sin^2\alpha} \qquad (0 \leqslant \alpha < \alpha_0) \tag{2.9}$$

$$\sum_{n=2}^{\infty} A_n (n + {}^1/_2)\,\mathrm{ch}\,(n + {}^1/_2)\,\beta_0 P_n^{\,2}(\cos\alpha) = 0 \quad (\alpha_0 < \alpha \leqslant \pi) \tag{2.10}$$

By setting

$$A_n \,\mathrm{ch}\left(n + \frac{1}{2}\right)\beta_0 = \int_0^{\alpha_0} \varphi(t) \cos\left(n + \frac{1}{2}\right) t \, dt \tag{2.11}$$

and taking into account the relation

$$P_n^{\,2}(\lambda) = (1 - \lambda^2)\frac{d^2}{d\lambda^2} P_n(\lambda), \qquad \lambda = \cos\alpha \tag{2.12}$$

as well as (1.12), (2.10) is identically satisfied. Taking note of (2.12), (1.11) and (1.18), and employing the integral representation

$$P_n(\cos\alpha) = \frac{\sqrt{2}}{\pi} \int_0^{\alpha} \frac{\cos(n + {}^1/_2)\,x}{\sqrt{\cos x - \cos\alpha}}\, dx \tag{2.13}$$

(2.9) is transformed into the form

$$\frac{d^2}{d\lambda^2} \int_0^{\alpha} \frac{dx}{\sqrt{\cos x - \cos\alpha}} \left\{ \varphi(x) - \frac{2}{\pi} \int_0^{\alpha_0} \varphi(t)\,[\eta(t + x) + \eta(t - x)]\, dt \right\} = C \frac{(1-\lambda)^{3/2}}{a^2 (1 - \lambda^2)^2} \tag{2.14}$$

Here, the function η is defined, as before, by (1.15).

Integrating (2.14) twice with respect to λ, we obtain Abel's integral Eq.

$$\int_0^{\alpha} \frac{dx}{\sqrt{\cos x - \cos\alpha}} \left\{ \varphi(x) - \frac{2}{\pi} \int_0^{\alpha_0} \varphi(t)\,[\eta(t + x) + \eta(t - x)]\, dt \right\} = \tag{2.15}$$

$$= -\frac{\sqrt{2}}{4} \frac{C}{a^2} \left[\sqrt{2}\sqrt{1-\lambda} + \frac{3}{2} \ln \frac{\sqrt{2} - \sqrt{1-\lambda}}{\sqrt{2} + \sqrt{1-\lambda}} + \lambda \ln \frac{\sqrt{1+\lambda}}{\sqrt{2} - \sqrt{1-\lambda}} \right] + c_1\lambda + c_2$$

Solving the preceding equation, we obtain, as in Section 1, a Fredholm integral equation for $\phi(x)$

$$\varphi(x) - \frac{2}{\pi} \int_0^{\alpha_0} \varphi(t)\,[\eta(t + x) + \eta(t - x)]\, dt =$$

$$= \frac{C}{a^2} \sin^3 \frac{x}{2} - \frac{\sqrt{2}}{\pi} c_1 (1 - 2\cos x) \cos\frac{x}{2} + \frac{\sqrt{2}}{\pi} c_2 \cos\frac{x}{2} \tag{2.16}$$

The constants c_1 and c_2 in (2.16) are obtained from the supplementary integrability condition for $\partial\psi/\partial s$ in the region $z = 0$, exterior to the punch (ψ is the displacement function [13]).

Note that this requirement represents the condition that the angle of twist of the punch ϑ be finite.

Utilizing the relation between ψ and Φ:

$$\frac{\partial\psi}{\partial s} = -\frac{1}{r^3}\frac{\partial\Phi}{\partial n} = -\frac{1}{r^3}\frac{\partial}{\partial n}(r^2 w) \tag{2.17}$$

we conclude that

$$\left.\frac{\partial\psi}{\partial s}\right|_{\beta=0} = -\left(\frac{1}{rh}\right)_{\beta=0}\left.\frac{\partial w}{\partial\beta}\right|_{\beta=0} \qquad \left(h = \frac{a}{\operatorname{ch}\beta - \cos\alpha}\right) \tag{2.18}$$

Here h is the Lamé coefficient.

Taking into account (2.4), (2.11), (2.12) and (1.12), we obtain after some manipulation

$$\left.\frac{\partial w}{\partial\beta}\right|_{\beta=0} = -\sqrt{1-\cos\alpha}\sin^2\alpha\frac{\Gamma d^2}{d(\cos\alpha)^2}\left[\frac{\varphi(\alpha_0)}{\sqrt{\cos\alpha-\cos\alpha_0}} - \int_\alpha^{\alpha_0}\frac{\varphi'(t)\,dt}{\sqrt{\cos\alpha-\cos t}}\right] \tag{2.19}$$

so that we must have

$$\varphi(\alpha_0) = 0 \tag{2.20}$$

Further, integrating (2.19) by parts, we obtain

$$\left.\frac{\partial w}{\partial\beta}\right|_{\beta=0} = \sqrt{1-\cos\alpha}\sin^2\alpha\frac{d^2}{d(\cos\alpha)^2}\left[2\sqrt{\cos\alpha-\cos\alpha_0}\frac{\varphi'(\alpha_0)}{\sin\alpha_0} - \right.$$

$$\left. - 2\int_\alpha^{\alpha_0}\left(\frac{\varphi'(t)}{\sin t}\right)_t'\sqrt{\cos\alpha-\cos t}\,dt\right] \tag{2.21}$$

From which it follows that:

$$\varphi'(\alpha_0) = 0 \tag{2.22}$$

Thus, (2.20) and (2.22) serve to determine c_1 and c_2, whence the formula for $\partial\psi/\partial s$ takes the form

$$\left.\frac{\partial\psi}{\partial s}\right|_{\beta=0} = \frac{2}{a^2}(1-\cos\alpha)^{5/2}\sin\alpha\frac{d^2}{d(\cos\alpha)^2}\int_\alpha^{\alpha_0}\left[\frac{\varphi'(t)}{\sin t}\right]_t'\sqrt{\cos\alpha-\cos t}\,dt \tag{2.23}$$

From (2.23), the relation between the angle of twist ϑ and the moment M is easily obtained. Indeed, integrating (2.23) in the region $\beta = 0$, $0 < \alpha < \alpha_0$ and taking into account the fact that $\psi \to 0$ at infinity, we obtain

$$\int_0^{\alpha_0}\frac{\partial\psi}{\partial s}ds = \vartheta \tag{2.24}$$

Substituting (2.23) into (2.24), we obtain after some manipulation

$$\vartheta = -\frac{3}{4}\frac{\pi}{a}\varphi(0) \tag{2.25}$$

The quantity $\phi(0)$ is proportional to the moment M, so that (2.25) is the desired relation.

To obtain an effective solution to this problem in a manner similar to that of Section 1, we expand the functions η and ϕ and the constants c_1 and c_2 in power series of $\varepsilon = e^{-\beta_0}$. Matching coefficients of like powers of in (2.16), we obtain for the zeroeth approximation

$$\varphi_0(x) = \frac{C}{a^2}\sin^3\frac{x}{2} - \frac{\sqrt{2}}{\pi}(1-2\cos x)\cos\frac{x}{2}c_{10} + \frac{\sqrt{2}}{\pi}\cos\frac{x}{2}c_{20} \tag{2.26}$$

Upon determining c_{10} and c_{20} from (2.20) and (2.22), we obtain the first relation in (1.28), which corresponds to the case of a half-space without a cavity. Additional computations yield

$$\varphi_k(x) = 0, \qquad k = 1-4 \tag{2.27}$$

so that the correction to the moment in this case is of fifth order of smallness.

The approximate formula for the moment M takes the form

$$M \approx M_0 (1 - \Delta), \qquad \Delta = \varepsilon^5 \frac{64}{\pi} \sin^6 \frac{\alpha_0}{2} \cos^3 \frac{\alpha_0}{2} \tag{2.28}$$

The coefficient $1 - \Delta$ indicates the decrease in the moment required in order to obtain an angle of twist ϑ for the half-space, taking into account the effect of a spherical cavity with a stress-free surface.

Values of Δ for certain values ρ/l and b/l are given below.

$\rho/l = 0.6$	0.7	0.8
$\Delta = 0.003$	0.012	0.038 ($b/l = 1/3$)
$\Delta = 0.005$	0.016	0.041 ($b/l = 1/2$)
$\Delta = 0.003$	0.008	0.014 ($b/l = 1$)

In conclusion, we note that the dual series method is also applicable to the case in which shearing stresses are applied to the cavity surface,

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